Mod-26 Arithmetic Disc

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`http://queue.acm.org/DrillBits`

Base-26 modular arithmetic by hand can be tedious and error-prone, particularly when several consecutive operations must be performed. A simple mechanical aid based on the next page can help. It consists of two nested 26-sided polygons; the inner polygon features dots around the periphery and a "clock hand" (the double line running from the center to the edge). Print the next page on paper, ideally thick cardstock. Arrange for the inner polygon to rotate freely within the fixed outer polygon by cutting along the dashed circle and placing an axle at the center, e.g., a thumbtack.

Clockwise rotation implements mod-26 addition: To sum two or more numbers, point the clock hand at A_0, representing an initial sum of zero. For each letter to be added to the sum, rotate the topmost dot on the inner polygon clockwise to the position of that letter on the outer polygon. For example, the first digit of the splitting-reconstruction example in my column is $T_{19}+D_3+O_{14}+X_{23} = H_7$. Follow these steps to compute the sum:

1. Start with the clock hand pointing upward, toward A_0.

2. Add T_{19} by rotating the topmost dot (the dot beneath A_0) clockwise to position T_{19}.

3. Add D_3 by rotating the topmost dot clockwise to position D_3.

4. Add O_{14} by rotating the topmost dot clockwise to position O_{14}.

5. Add X_{23} by rotating the topmost dot clockwise to position X_{23}.

6. The clock hand points to the sum, H_7.

Counter-clockwise rotation implements subtraction. For example, the first digit of the "HELLO $-$ WUJYD $=$ LKCNL" example in my column is $H_7 - W_{22} = L_{11}$. Follow these steps to compute the difference:

1. Start with the clock hand pointing toward H_7.

2. Subtract W_{22} by rotating the dot *at position* W_{22} counter-clockwise to the topmost position.

3. The clock hand points to the difference, L_{11}.

After a while you'll realize that the direction of rotation doesn't really matter in practice. The important thing is that addition rotates the dot *at the top* into the position of the letter to be added, and subtraction rotates the dot *at the position of the letter to be subtracted* to the top.